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INAUGURAL LECTURE

MINIMISATION OF ERROR: A NECESSARY CONDITION FOR ACCURACY, STABILITY AND PROGRESS

BY

PROFESSOR RAPHAEL BABATUNDE ADENIYI
B.Sc; M.Sc, Ph.D. (Ilorin)
DEPARTMENT OF MATHEMATICS
FACULTY OF PHYSICAL SCIENCES
UNIVERSITY OF ILORIN, ILORIN, NIGERIA

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The Vice-Chancellor

Professor AbdulGaniyu Ambali (OON)
DVM (Zaria), M.V.Sc, Ph.D. (Liverpool), MCVSN (Abuja)

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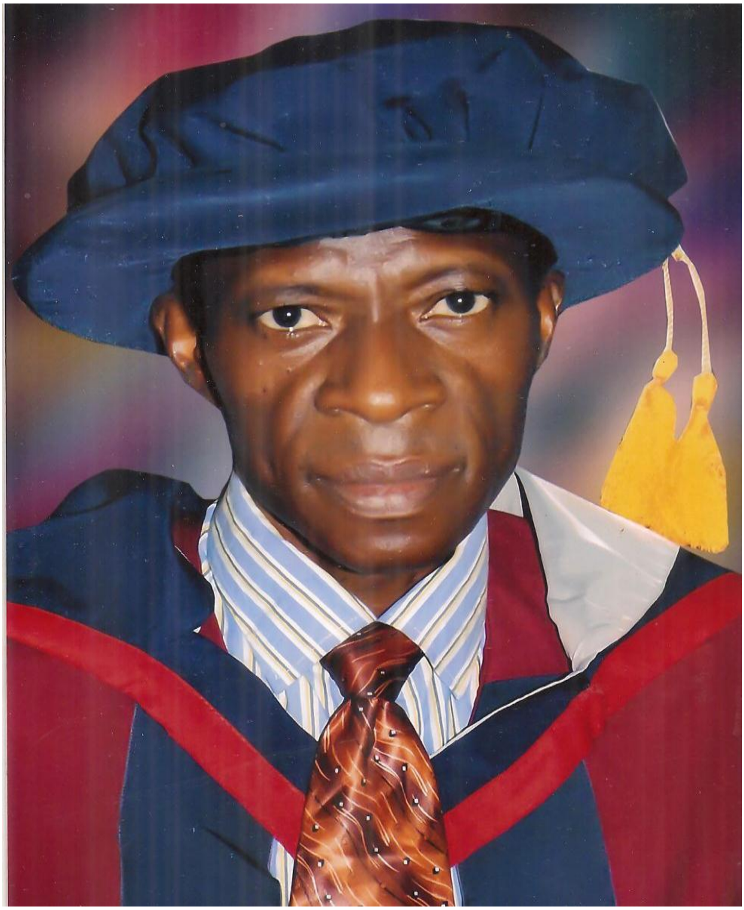

PROFESSOR RAPHAEL BABATUNDE ADENIYI
B.Sc.; M.Sc.; Ph.D. (Ilorin)
PROFESSOR OF MATHEMATICS

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My Lords, Spiritual and Temporal,
Distinguished Students of Mathematics,
Gentlemen of the Print and Electronic Media,
Friends and Relations,
Distinguished Invited Guests,
Great Unilorites,
Ladies and Gentlemen.

1.0 Preamble

All adoration, dominion, power and majesty be unto the Most High Lord, GOD Almighty for the privilege to deliver the 162nd Inaugural Lecture of the University of Ilorin to this distinguished august assembly. This is the third of such lectures coming from the Department of Mathematics. By the Grace of God, this is the first from the latest crop of Professors in this University and also the first by a Numerical Analyst in this

University. Even before I received my letter of appointment (that is, the period between the meeting of the University's Appointment and Promotion Committee and that of the Governing Council), I was somewhat inundated with calls and requests from several quarters as to when I would deliver my Inaugural Lecture. This may not be unexpected as the reason is not farfetched. According to the Holy Bible "hope deferred makes the heart sick but when the desire comes it is a tree of life". However for this feat, though somehow circumstantial, I am most grateful to the Lord GOD and for this opportunity I thank the Vice-Chancellor and the University Administration.

1.1 Becoming A Mathematician

Mr. Vice-Chancellor sir, my being a Mathematician was not by accident. After the completion of my Advanced Level Course in the then Kwara State College of Technology, I applied to University of Nsukka to study Architecture or Estate Management. However, I was not offered an admission that year. In order not to miss admission into a University the following year, I applied to the University of Ilorin to read Mathematics/Education which I believed was not a competitive course (at least as at that time) in terms of admission. But, as GOD would have it, I was admitted for a single honours degree in Mathematics. I am most grateful to Him (again) because I did not have any cause to regret my coming to this presently most subscribed University in Nigeria by prospective applicants, to read Mathematics.

2.0 Introduction

This lecture mainly focuses on two numerical integration techniques for solution of Differential Equations (DEs). These are the tau method and linear multistep methods. While the tau method was originally conceived as a continuous scheme, the linear multistep methods originated as discrete schemes. However, both methods have crossed each other's original boundary so that the tau method can now be formulated as discrete scheme and the formulation of multistep method as continuous scheme is now feasible. My research on the tau method is concerned with its error analysis by which I was able to accurately estimate the error. With regard to the linear multistep methods, my focus was directed at developing continuous schemes both for existing methods and new schemes. As will be discussed later, these two areas are important aspects of Numerical Analysis and are of significant import in Mathematics and more especially in Computational Mathematics.

Mathematics is the study of numbers, quantity, space, structure and change. It is used as an essential tool in many fields including natural science, engineering, medicine and the social sciences. Applied mathematics which is concerned with application of knowledge of Mathematics to other fields, inspires and makes use of new mathematical discoveries and sometimes leads to the development of entirely new mathematical disciplines such as Statistics and Game theory. Mathematicians also engage in pure mathematics or mathematics for its own sake, without having any application in mind. There is no clear line of separating pure and applied mathematics, and practical applications for what began as pure mathematics may require some computations. Thus, for example, Numerical Linear Algebra has evolved from Linear Algebra.

Numerical analysis is the study of algorithms that use numerical approximations for the problems of mathematical analysis. It is concerned with obtaining approximate solutions while maintaining reasonable bounds on error. Numerical analysis naturally finds applications in all fields of engineering and physical sciences, but in the 21st century, the life sciences and even humanities have adopted elements of scientific computations. Also, numerical analysis is the area of mathematics and computer science that creates, analyses and implements algorithms for obtaining numerical solutions to problems involving continuous variables. Such problems arise throughout the natural sciences, social sciences and business. Since the mid-20th century, the growth in power and availability of digital computers has led to an increasing use of realistic mathematical models in science and engineering and numerical analysis of increasing sophistication is needed to solve these more detailed models of the world. The formal academic area of numerical analysis ranges from quite theoretical mathematical studies to computer science.

Put succinctly, numerical analysis is concerned with all aspects of the numerical solution of a problem, from the theoretical development and understanding of numerical methods to their practical implementation as reliable and efficient computer programs. The need for numerical analysis arose as a result of non-availability of analytic methods for solving all mathematical problems. Since numerical methods often provide approximate solutions to mathematical problems, the subject of error must necessarily come into play. Error, in this wise, is the difference between the exact (expected or desired actual) solution and the approximate (computed) solution resulting from the use of numerical methods. Four types of errors may naturally result from numerical computations namely, round – off error, truncation error, human error (or blunder) and inherent error.

Many numbers resulting from the use of computing tools have infinite decimal representations and hence such numbers may have to be rounded up to some significant figures or decimal places of accuracy as desirable. Also infinite processes may need to be truncated to the desired number of terms for practical use. This results into truncation error. Blunders originate from the person who implements a numerical algorithm. An example is the transposition of numbers whereby the position of two figures in a given number may be mistakenly interchanged. Sometimes input data may also contain some error arising from the conduct of experiments, thus leading to inherent error. The study of error is thus of central concern in numerical analysis, otherwise the technique adopted will just end up as a numerical method. Permit me, therefore, to use the crude equation

Numerical method + error analysis = Numerical analysis

Although numerical methods are important in solving mathematical problems where analytic methods are not available, the exclusion of a procedure to analyse, or bound or estimate its error may make the numerical scheme incomplete or undesirable.

In this treatise my focus is firstly on the error estimation of the tau method. However, before proceeding to the tau method, let me briefly say something about the 'catalyst' that makes the tau method attractive and that is the Chebyshev polynomial.

3. The Chebyshev Polynomials

The problem of approximating a function is of great significance in numerical analysis due to its importance in the development of software for digital computers. Function evaluation through polynomial interpolation techniques over stored table of values and which derives its justification from the Weierstrass theorem, has been found to be quite costlier when compared to the use of efficient function approximations. If $f_1, f_2, f_3, \dots . f_n$ are the values of the given function and $\varphi_1, \varphi_2, \varphi_3, \dots, \varphi_n$ are the corresponding values of the approximating function, then the error vector $\underline{e}$ is given by

$$e_k = f_k - \varphi_k$$

for k=1(1)n. The approximating function may be chosen in a number of ways. For example, we may find the approximating function such that the square root of the sum of the square of the components of the error function is minimum. This is the idea of Least squares approximation. On the other hand, the approximating function may be chosen such that the maximum component of the error vector is minimised. This leads to the Chebyshev polynomials which have found important applications in the approximations of functions in digital computers.

The Chebyshev polynomial of the first kind is defined by

$$T_n(x) = cos(ncos^{-1}x), -1 \leq x \leq 1 \qquad (3.1)$$

and it satisfies the triple recursion relation

$$T_n(x) = 2xT_n(x) - T_{n-1}(x), \quad n \geq 1 \ . \qquad (3.2)$$

The latter formula is often used in generating these polynomials rather than the former (explicit) form (3.1). It is to be noted that

$$|T_n(x)| \leq 1, -1 \leq x \leq 1 \qquad (3.3)$$

and that

$$T_n(x) = 2^{n-1}x^n + lower\ degree\ terms.$$

From this we get the monomial (a polynomial whose leading coefficient is unity)

$$\widetilde{T_n}(x) = \frac{1}{2^{n-1}}T_n(x) = \ x^n + lower\ degree\ terms \qquad (3.4)$$

so that

$$\left|\widetilde{T_n}(x)\right| \leq \frac{1}{2^{n-1}} \ , -1 \leq x \leq 1 \ . \qquad (3.5)$$

Of all monomials, $p_n(x)$, the polynomial $\widetilde{T_n}(x)$ of equation (3.4) has the smallest least upper bound for its absolute value in the interval (-1, 1), and this upper bound is $\frac{1}{2^{n-1}}$ (from equation (3.5)). Thus, in Chebyshev approximation, the maximum error is kept down to a minimum and this is often referred to as the Mini-max principle and $\widetilde{T_n}(x)$ is called the Mini-max polynomial.

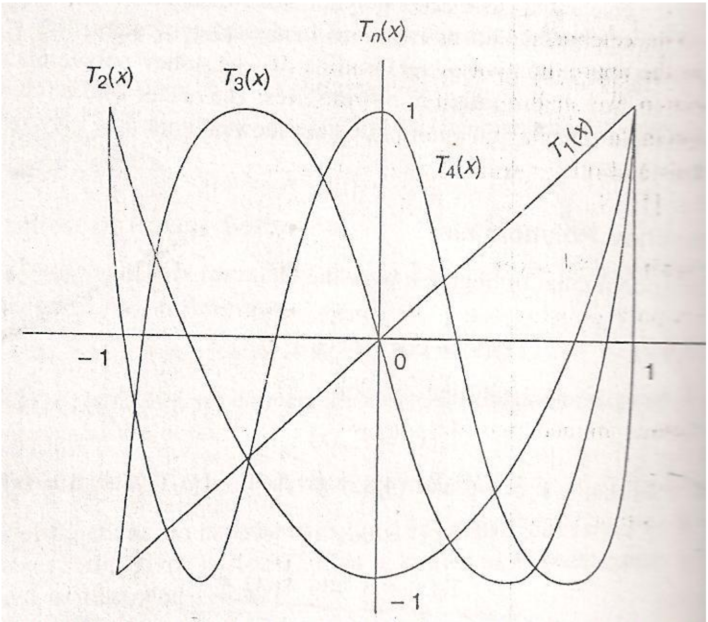

Figure 1. Chebyshev Polynomial $T_n(x), -1 \leq x \leq 1$

The Chebyshev polynomial also oscillates with equal amplitude in its entire range of definition (see Figure 1 above) and thus ensuring equal distribution of error throughout its range of definition. This is in contra- distinction to the popular Taylor polynomial that only guarantees minimum error at the origin but deviates more and more as one moves away from this origin.

4.0 **The Tau Method**

Accurate approximate solution of Initial Value Problems (IVPs) and Boundary Value Problems (BVPs) in linear Ordinary Differential Equations (ODEs) with polynomial coefficients can be obtained by the tau method originally introduced by Lanczos (1938). Techniques based on this method have been reported in literature with applications to more general equations including non-linear ones (Ortiz, 1969 and Onumanyi and Ortiz, 1982) while techniques based on direct Chebyshev series replacement have been discussed by Fox (1962) as well as to both partial differential equations and integral equations (Adeniyi, 2004). The tau method in its most three important variants is as follows:

4.1 Differential Formulation of the Tau Method

Consider the m[th] order ordinary differential equation

$$Ly(x) := \sum_{r=0}^{m} P_r(x) y^{(r)}(x) = f(x), \quad a \le x \le b \qquad (4.1a)$$

with associated conditions

$$L^* y(x_{rk}) := \sum_{r=0}^{m} a_{rk} y^{(r)}(x_{rk}) = \alpha_k, \quad k = 1(1)m \qquad (4.1b)$$

and where $|a| < \infty$, $|b| < \infty$, a_{rk}, x_{rk}, α_k, $r = 0(1)m\text{-}1$, $k = 1(1)m$, are given real numbers; $f(x)$ and $P_r(x)$, $r = 0(1)m$, are polynomial functions or sufficiently close polynomials approximants of given real functions.

For the solution of problem (4.1) by the Tau method, we seek an approximant of the form

$$y_n(x) = \sum_{r=0}^{n} a_r x^r, \quad n < +\infty \qquad (4.2)$$

of y(x) which satisfies exactly the perturbed problem:

$$Ly_n(x) = f(x) + \sum_{r=0}^{m+s-1} \tau_{m+s-r} T_{n-m+r+1}(x) \qquad (4.3a)$$

$$L^* y_n(x_{rk}) = \alpha_k, \quad k = 1(1)m \qquad (4.3b)$$

for a ≤ x ≤ b and where τ_r, $r = 1(1)m+s$, are parameters to be determined along with the coefficients a_r, $r = 0(1)m$, in (4.2).

$$T_r(x) = \cos\left\{ r \cos^{-1}\left[\frac{(2x-2a)}{(b-a)} - 1 \right] \right\} \equiv \sum_{k=0}^{r} C_k^{(r)} x^k \qquad (4.4)$$

is the r-th degree Chebyshev polynomial valid in the interval [a,b] and

$$s = \max\{|N_r - r|:\ 0 \le r \le m\}$$

is the number of overdetermination of equation (4.1a) (see Fox, 1968).

We determine a_r, $r = 0(1)n$, and τ_r , $r = 1(1)m+s$, by equating corresponding coefficients of power of x in equation (4.3a) together with conditions (4.3b). Consequently, we obtain the desired approximant $y_n(x)$ in (4.2).

4.2 The Integral Formulation of the Tau Method

The integrated form of equation (4.1a) is given by

$$I_L(y(x)) = \underbrace{\iiint}_{\ldots} m \int\ f(x)dx + C_m(x) \qquad (4.6)$$

where $C_m(x)$ denotes an arbitrary polynomial of degree $(m\text{-}1)$, arising from the constants of integration and

$$I_L = \underbrace{\iiint}_{\ldots} m \int\ L(\cdot)dx, \qquad (4.7)$$

is the m-times indefinite integration of L(·). The corresponding tau problem is therefore:

$$I_L(\omega_n(x)) = \underbrace{\iiint}_{\ldots} m \int\ f(x)dx + C_m(x) + \sum_{r=0}^{m+s-1} \tau_{m+s-1} T_{n+r+1}(x) \qquad (4.8a)$$

$$L^* \omega_n(x_{rk}) = \alpha_k, \quad k = 1(1)m \qquad (4.8b)$$

where

$$\omega_n(x) = \sum_{r=0}^{n} b_r x^r, \quad n < +\infty \quad . \qquad (4.9a)$$

The b_r's are constants to be determined along with the τ_r 's in equation (4.8a) . Problem (4.8) often gives a more accurate tau approximant than equation (4.3) does, due to its higher order perturbation term.

4.3 The Recursive Formulation of the Tau Method

The so-called canonical polynomials $\{Q_r(x)\}$, $r \varepsilon N_0\text{-}S$, associated with the operator of equation (4.1) is defined by

$$LQ_r(x) = x^r \tag{4.9b}$$

where S is a small finite or empty set of indices with cardinality $s(s \leq m + h)$, h being the maximum difference between the exponent of the generating polynomial Lx^r, *for* $r \varepsilon N_0$. (see Ortiz, 1969 and 1974). Once these polynomials are generated, we seek, in this case, an approximant of $y(x)$ of the form

$$v_n(x) = \sum_{r=0}^{n} d_r Q_r(x) \tag{4.10}$$

which is identically given by

$$v_n(x) = \sum_{r=0}^{F} f_r Q_r(x) + \sum_{r=0}^{m+s-1} \tau_{m+s-r} \sum_{k=0}^{n-m+r+1} C_k^{(n-m+v+1)} Q_k(x) \tag{4.11}$$

and where f_r, $r = 0(1)F$, are the coefficients in $f(x)$ and the d_r's are constants to be determined along with the τ_r 's in equation (4.11). The use of the Q_r's is advantageous as they neither depend on the boundary conditions nor on the interval of solution. Furthermore, they are re-usable for approximants of higher degree.

5. Error Analysis of Tau Method

The first attempt on an error estimation of the tau method was by Lanczos (1956) where he developed a simple algebraic approach to this problem by using the relation of the Chebyshev polynomials to trigonometric function, and which was applied only to the restricted class of first order problems:

$$A(x)y'(x)+B(x)y(x)+C(x)=0, \; y(x_0)=y_0, \; 0 \leq x \leq 1 \; . \tag{5.1}$$

The coefficients A,B and C are polynomial functions. Fox (1968) later developed an approach which could handle higher order problems. However, his approach was not general in application. Namasivayam and Ortiz (1981) deduced asymptotic estimates for the tau method approximation error vector per step for different choices of perturbation term. Crisci and Ortiz (1981) reported the existence and convergence results for the numerical solution of differential equations with the tau method. Freilich and Ortiz (1982) obtained bounds for the error of the recursive formulation of the tau method when applied to system of ODEs while Freilich and Ortiz (1991) showed that the error analysis for a rational tau method can be combined to give upper and lower error bounds for the error vector of the tau method for rational approximations. Also, Namasivayam and Ortiz (1993) reported the dependence of the error approximation of the tau method on the choice of perturbation terms.

The spring board for my research on the subject of error analysis of the tau method was the work of Onumanyi and Ortiz (1982). This practical error estimation approach therefore deserves greater attention; the approach follows a similar trend as the tau method itself and so involves the determination of several unknown coefficients of an assumed error polynomial function just in the same way as the coefficients of the tau polynomial approximation of Section 4 are determined. Although the error estimate in this case was accurate, the method is not considered efficient because of the several parameters involved. It is necessary to remark here that Onumanyi (1983) also reported a more efficient but less than general error estimation of the tau method.

6. My Humble Contributions

The motivation for my research activities on the subject of the error analysis of the tau method was derived from the fact that the approximate solution $y_n(x)$ to the analytical (exact) solution of an ODE by the tau method as proposed by Lanczos (1938) is an economized polynomial function implicitly defined by the ODE. Consequently, the error of tau method could be estimated by the process of economization of a power series. This process is a procedure by which the number of terms of the truncated series representation of a given function defined over an interval can be reduced without substantially damaging the accuracy of the original function over that interval.

To economize the n^th degree polynomial approximation

$$p_n(x) = \sum_{r=0}^{n} a_r x^r \cong f(x) = \sum_{r=0}^{\infty} a_r x^r \qquad (6.1)$$

of a given function $f(x)$, the best choice of polynomial to employ is the Chebyshev polynomial $T_n(x)$ of Section 3 appropriately defined in the range of definition of $f(x)$.

This is because (as mentioned earlier), the monomial

$$\frac{T_n(x)}{C_n^{(n)}} = x^n + \frac{1}{C_n^{(n)}} \sum_{r=0}^{n-1} C_r^n x^r \qquad (6.2)$$

has a small upper bound to its magnitude in that interval than any other monomial. As the maximum magnitude of $T_n(x)$ is unity, the upper bound thus referred to is $\left(C_n^{(n)}\right)^{-1}$.

Hence, to economize the approximation (6.1) we have that

$$P_n(x) = \sum_{r=0}^{n-1} a_r x^r + a_n x^n = \sum_{r=0}^{n-1} a_r x^r + a_n \left(\frac{T_n(x)}{C_n^{(n)}} - \frac{1}{C_n^{(n)}} \sum_{r=0}^{n-1} C_r^n x^r \right).$$

That is,

$$P_n(x) = \sum_{r=0}^{n-1} \bar{a}_r x^r + \frac{a_n T_n(x)}{C_n^{(n)}} \quad . \qquad (6.3)$$

So then, the error in the economization of (6.1) to a polynomial of degree n-1 is the function

$$\eta(x) = \frac{a_n T_n(x)}{C_n^{(n)}} \qquad (6.4)$$

Consequently, the error estimation I have proposed was based on a modification of equation (6.4). I modified equation (6.4) to have the error polynomial function

$$((e_n(x))_{m+1} = \frac{\varphi_n \mu_m(x) T_{n-m+1}(x)}{C_{n-m+1}^{(n-m+1)}} \qquad (6.5)$$

where φ_n is a parameter to be determined and $\mu_m(x)$ is an m[th] degree polynomial function chosen to ensure that the m conditions associated with an m[th] order ODE are satisfied. For an IVP whose conditions are all specified at only one point α say, the function takes the form

$$\mu_m(x) = (x-\alpha)^m \qquad (6.6)$$

From my study, it was observed that this choice could also be appropriate for BVPs, in which case we assumed that some of the homogeneous conditions of the error function

$$e_n(x) = y(x) - y_n(x) \cong (e_n(x))_{m+1} \qquad (6.7)$$

are perturbed. This can lead to increase in accuracy of the error estimate of the tau method as also suggested by Fox and Parker (1968). This is illustrated by the two examples in the next section.

7.0 Error Estimation of the Method

My error estimation of the tau method for the three variants described above is now briefly described here.

7.1 Error Estimation for the Differential Form

While the error function for the differential form of the tau method

$$e_{n,D}(x) = y(x) - y_n(x) \qquad (7.1)$$

satisfies the error problem

$$Le_{n,D}(x) = -\sum_{r=0}^{m+s+1} \tau_{m+s-r} T_{n-m+r+1}(x) \qquad (7.2a)$$

$$L^* e_{n,D}(x_{rk}) = 0, \quad k = 0(1)m, \qquad (7.2b)$$

the error approximant

$$\left(e_{n,D}(x)\right)_{n+1} = \mu_m(x)\varphi_{n,D}\, T_{n-m+1}(x) \Big/ C_{n-m+1}^{(n-m+1)} \qquad (7.3)$$

satisfies the perturbed error problem

$$L\left(e_{n,D}(x)\right)_{n+1} = -\sum_{r=0}^{m+s+1} \tau_{m+s-r} T_{n-m+r+1}(x) + \sum_{r=0}^{m+s+1} \tilde{\tau}_{m+s-r} T_{n-m+r+2}(x) \qquad (7.4a)$$

$$L^*\left(e_{n,D}(x)\right)_{n+1} = 0 \qquad (7.4b)$$

where the extra parameters $\tilde{\tau}_r$, $r = 1(1)m+s$, and $\varphi_{n,D}$ are to be determined and $\mu_m(x)$ is a specified polynomial of degree m which ensures that $\left(e_{n,D}(x)\right)_{n+1}$ satisfies the homogeneous conditions in equation (7.2b). We insert (7.3) in equation (7.4a) and then equate corresponding coefficients of x^{m+s+1}, x^{n+s}, …, x^{n-m+1} and the resulting linear system is solved for only $\varphi_{n,D}$ by forward elimination, since we do not need the $\tilde{\tau}_r$'s in equation (7.4a). Consequently,

$$\begin{aligned}\bar{\varepsilon}_D &= \max_{a\le x\le b}\left|\left(e_{n,D}(x)\right)_{n+1}\right| = \left|\varphi_{n,D}\right| \Big/ \left|C_{n-m+1}^{(n-m+1)}\right| \\ &= 2^{2n-2m+1}\left|R_D^{-1}\right| \left|b-a\right|^{2m-n-1}\left|\tau_{m+s}\right| \\ &\cong \max_{a\le x\le b}\left|e_{n,D}(x)\right| = \varepsilon_D \end{aligned} \qquad (7.5)$$

where R_D is to be determined. (Details can be found in Adeniyi *et al*, 1991).

7.2 Error Estimation for the Integral Form

The error polynomial

$$\left(e_{n,I}(x)\right)_{n+1} = \mu_m(x)\varphi_{n,I}\, T_{n-m+1}(x) \Big/ C_{n-m+1}^{(n-m+1)} \qquad (7.6)$$

satisfies the integrated perturbed error problem:

$$I_L(e_{n,I}(x)_{n+1} = -\iint\limits_{\cdots} m \int \left(\sum_{r=0}^{m+s-1} \tau_{m+s-r} T_{n-m+r+1}(x) \right) dx + C_m(x)$$
$$+ \sum_{r=0}^{m+s-1} \bar{\tau}_{m+s-r} T_{n-m+r+3}(x) \qquad .(7.7)$$

We insert (7.6) in equation (7.7) and then equate coefficients of $x^{n+s+m+1}$, x^{n+s+m}, …, x^{n-m} for the determination of the parameter $\varphi_{n,I}$ in (7.6). Subsequent procedures follow as described above in Section 7.1 in order to obtain the error estimate:

$$\bar{\varepsilon}_I = \max_{a \le x \le b} \left| \left(e_{n,I}(x)\right)_{n+1} \right| = \left|\varphi_{n,I}\right| \Big/ \left| C_{n-m+1}^{(n-m+1)} \right|$$
$$\cong \max_{a \le x \le b} \left| e_{n,I}(x) \right| = \varepsilon_I \,. \qquad (7.8)$$

7.3 Error Estimation for the Recursive Form

Once the canonical polynomials of Sub-section 4.3 are generated, they can be used for an error estimation of the tau method (see Crisci and Ortiz, 1981 and Namsivayam and Ortiz, 1981). Here we considered a slight perturbation of the given boundary conditions (4.1b) by $\bar{\varepsilon}_D$ to obtain an estimate of the tau parameter τ_{m+s}, in terms of canonical polynomials, which is then substituted back into the expression for $\bar{\varepsilon}_D$ in equation (7.5) for a new estimate $\bar{\varepsilon}_R$.

So doing, we have from (4.1b) that

$$\left|\sum_{r=0}^{m} a_{rk} y^{(r)}(x_{rk})\right| \leqq |\alpha_k| + \bar{\varepsilon}_D, \quad (k=m)$$
$$= |\alpha_m| + 2^{2n-2m+1}\left|R_D^{-1}\right| \, |b-a|^{2m-n-1}|\tau_{m+s}|$$

since $\bar{\varepsilon}_D \geqq 0$. This leads to the inequality

$$|\beta||\tau_{m+s}| \leqq |\alpha_m||\eta| + 2^{2n-2m+1}\left|R_D^{-1}\right| \, |b-a|^{2m-n-1}|\tau_{m+s}| .$$

The two quantities β and η are expressions which depend on the canonical polynomials and the derivatives of these polynomials when evaluated at some points of [a,b].

Thus, we have

$$\left[|\beta| - 2^{2n-2m+1}\left|R_D^{-1}\right| \, |b-a|^{2m-n-1}\right]|\tau_{m+s}| \leqq |\alpha_m||\eta| ,$$

giving us

$$|\tau_{m+s}| \leqq \frac{|\alpha_m||\eta||R_D|}{|\beta| \, |R_D| - 2^{2n-2m+1} \, |b-a|^{2m-n-1}} . \qquad (7.9)$$

So, from equation (7.5), we have

$$\bar{\varepsilon}_D \leqq \frac{2^{2n-2m+1} \, |b-a|^{2m-n-1}}{|R_D|}\left[\frac{|\alpha_m||\eta||R_D|}{|\beta| \, |R_D| - 2^{2n-2m+1} \, |b-a|^{2m-n-1}}\right]$$
$$= \frac{|\alpha_m||\eta|}{2^{2m-2n+1} \, |b-a|^{n-2m+1}|\beta| \, |R_D| - 1} = \bar{\varepsilon}_R .$$

Thus, we have, as our error estimate,

$$\bar{\varepsilon}_R = \frac{|\alpha_m||\eta|}{2^{2n-2n+1}\ |b-a|^{n-2m+1}|\eta|\ |R_D|-1}$$
$$\cong \max_{a\le x\le b}|e_{n,R}(x)| = \varepsilon_R \qquad (7.10)$$

Where

$$e_{n,R}(x) = y(x) - V_n(x).$$

For the purpose of automation of the tau method with its error estimate, a generalised approach to the subject is most desirable. Consequently, Adeniyi *et al.* (1991) obtained the error estimate of the differential form for the general class of problem (4.1). The general formula thus obtained:

$$\varepsilon = \max_x\{|e_n(x)|\} \cong \max_x\{|(e_n(x))_{m+1}|\} \le \frac{\left[2^{2n-2m+1}(b-a)^{2m-n-1}|\tau_{m+s}|\right]}{|R_{m+s+1}|} = \bar{\varepsilon}$$

(7.11)

where R_{m+s+1} is given recursively by

$$R_1 = \lambda_1$$

$$R_v = \lambda_v - \sum_{k=1}^{v-1}\frac{C_{n+s+2-v}^{(n+s+2-k)}}{C_{n+s+2-v}^{(n+s+2-k)}}R_k;\ v = 2,3,\cdots,m+s+1$$

and

$$\lambda_1 = \sum_{r=1}^{v}\left(\sum_{k=0k}^{m}p_{k,s-v+r+k}\right)(k!)\binom{n+2-r}{k}D_r;$$

$$D_r = \begin{cases} C_{n-m-v+2} + \sum_{r=1}^{v-1}(-1)^r\binom{m}{r}\alpha^r C_{n-m-v+2}^{(n-m+1)}\ ;;\ v = 1,2,\cdots,m \\ C_{n-m-v+2}^{(n-m+1)} + \sum_{r=1}^{m}(-1)^r\binom{m}{r}\alpha^r C_{n-m-v+r+2}^{(n-m+1)};\ v = m+1,\cdots,m+s+1 \end{cases}$$

was itself partly recursive and is fast and reliable. This led to the development of a computer programme which could effectively and efficiently handle the problem for every member of class (4.1).

The scope of application of estimate (7.30) was extended in Adeniyi *et al*. (1991) to nonlinear problems through the process of linearization by the Newton-Kantorovich process. Once all information about a given DE is supplied, the programme would “output” the tau approximation, the corresponding exact error and the associated error estimate.

For problems with non-smooth solution in the range of definition, the partitioning of the range is necessary so that subsequently a segmented approach of the tau method could be adopted. This was the focus in Adeniyi (1993) where the error estimate for the piece-wise tau method was reported. The results showed that, as the uniform step-length of the segments decreased, the accuracy of the error estimate improved also just as the degree of tau approximation increased.

In 1982, Crisci and Russo reported an analysis of the stability of a one-step integration scheme which was originated from the Lanczos tau method and applicable to IVPs in first order linear ODEs. This method which was based on the canonical polynomials was of discrete form. The desire to extend the scope of my error estimation technique led me to consider this class of problems as reported in Adeniyi (1996). Again, the error estimate obtained was good as it gave, accurately, the order of the tau approximant being sought. A pertinent question was then that: is the error estimate obtained accurately optimum? This was the issue I addressed in Adeniyi (2000) where the optimality of the error estimate for the one-step discrete tau method was studied. It was confirmed the estimate was optimum. Thus, as in the cases for the continuous forms of the tau method, the error estimate for the discrete form was also accurate and effective.

Having generalised the error estimation for the differential form as described above, the concern in Yisa and

Adeniyi (2015) and Issa and Adeniyi (2013) was that: could it be done for the recursive form of the tau method? The answer was in the affirmative. This subsequently led us to an extension of this general recursive form to nonlinear problems in Issa and Adeniyi (2013). For an all-encompassing work on the subject of error estimation of the tau method for ODEs, it was needful to carry out the same investigation for the integral form/variant. This was the focus in Ma'ali *et al.* (2014)

Differential equations are mainly of two types namely ODEs and PDEs. All the works reported above related to ODEs. The need to further extend my study on error estimation to partial differential equations led to our works in Adeniyi and Erebholo (2007) where an analogue of the tau method for solving initial and boundary value problems was considered, in Biala; and Adeniyi (2015) where we combined the method of lines and the tau method for a "line-tau" collocation method; and in Erebholo and Adeniyi (2015) where a prior integration technique was the focus. A trend which I consistently observed in all these research on the subject of error estimation of the tau method was that for linear problems, nonlinear problems, piece tau approximation problems in ODEs, and also for PDEs, the error estimates for all the variants closely captured the order of the tau approximation and improved significantly as the degree of the approximation increased.

8.0 Development of Continuous Formulation of Finite Difference Schemes

The focus here was to develop continuous schemes/methods for the solution of the class of problems

$$y^{(k)}(x) = f\left(x, y, y', \ldots, y^{(k-1)}\right), \qquad (8.1a)$$

$$y^{(r)}(x_0) = y_r, r = 0,1, \ldots, k-1 \qquad (8.1b)$$

for k=1,2,3,4.

The numerical solution of ODEs by collocation methods has been well studied (see Lambert, 1973; Zennaro, 1985 and Fairweather and Meade, 1989). In particular, Wright (1970) established some relationships between certain Runge-Kutta methods and one-step collocation methods. Sarafyan (1990) provided algorithms for continuous solutions by Runge-Kutta methods with computational advantages. Lie and Norsett (1989) developed a multistep collocation method which showed that the Backward Differentiation Formulae (BDF) and the one-leg methods of Dahlquist can be produced from their formulation if collocation is done at one point.

In 1993 I, together with my colleagues, proposed a power series approach to multistep collocation which produced, for the first order equation, the Gragg-Stetter method of order four, the Hammer and Hollingworth method of order four, the BDF – methods, the Adam-Bashforth and Adam-Moulton methods, the optimal k-step methods and the Mid-point method. In addition, we derived a new class of accurate k-step methods *($k \geq 2$)* with adequate stability intervals for non-stiff problems. For the second order equation without first derivative present, the Numerov method of order four was produced by collocation.

This early work on the subject of development of continuous formulation of multistep methods through the process of collocation is presented here also for interested members of this audience.

Consider the IVP

$$y'(x) = f(x, y), y(a) = y_0 \quad , \quad a \leq x \leq b \qquad (8.2)$$

We assumed that problem (8.2) has a unique smooth solution $y \, \varepsilon \, \mathbb{R}^n$, $f \, \varepsilon \, \mathbb{R}^n$ and $a = x_0 < x_1 \ldots\ldots < x_N = b$

We assumed further, a constant step size $h = x_{i+1} - x_i$ and adopted a notation

$\bar{y}(x_0) = y_0 \quad , \quad \bar{y}(x_{i+j}) = y_{i+j} \quad j = 1 , 2 , \ldots ; k$ being a step number.

8.1 The Collocation Method

We assumed an approximate solution of the form

$$\bar{y}(x) = \sum_{r=0}^{n} a_r \ x^r \quad , \quad n = M + I - 1 \qquad (8.2)$$

where $x \ \varepsilon [x_i \ , \ x_{1+k}]$, $M \geq 1$ denotes the number of points used and $I \geq 1$ denotes the number of interpolation points used.

There are *n + 1* necessary equations needed to be used in determining the unique values of $a_0, \ a_1, \ldots \ n$ in equation (8.2). These equations are given by a selection from

$$\bar{y}(x_{1+j}) = y_{1+j} \ ; \ j = 0 \ \ldots , \ k = 1 \qquad (8.3)$$

$$\bar{y}'(x_j) = f\left(\bar{x}_j \ , y(x_j)\right); \quad j = 1, 2, \ldots M \qquad (8.4)$$

where $\bar{x} \, \varepsilon \{x_i \ , \ x_{i+1}, \ldots, \ x_{i+k}\} \cup (x_{i+k-1} \ , \ x_{i+k})$

The collocation method can be achieved step by step as follows:

(a) specify k;

(b) select the required *(n + 1)* equations from (8.3) and (8.4);

(c) solved by Gaussian elimination method the (n + 1) equations for $a_0, \ \ldots \ a_n$;

(d) obtain $\bar{y}(x)$ and $\bar{y}(x_{i+1})$.

Remark

The order of the collocation method is P = n + m where m $\leq$ M denotes the number of collocation points at the Gaussian points. If all collocation points are at the Gaussian points then P = I + $2m$ -1 , and if I = k then $P = k + 2m - 1$.

8.2 Collocation At The Off-Grid Points

Collocation at the off-grid points x_{i+k-1} , x_{i+k} is considered in this section.

8.2.1 **Gaussian Point Method (k = 1)**

Case n = 1

From (8.3) and (8.4), we select the following equations

$$\bar{y}(x_i) = \bar{y}_i; \; I = 1$$

$$\bar{y}^1\left(x_{i+\frac{1}{2}}\right) = f_{i+\frac{1}{2}}, \qquad M = 1$$

to obtain the values of α_0 and α_1 in equation (8.2), where $x_{i+\frac{1}{2}} = \frac{1}{2}(x_i + x_{i+1})$ is a Gaussian point in $[x_i \;, x_{i+l}]$. The resulting approximations after simplification are:

$$\left.\begin{aligned} \bar{y}(x) &= y_i + (x - x_i)\, f_{i+\frac{1}{2}} \\ f_{i+\frac{1}{2}} &\equiv f\left(x_j + \frac{1}{2}h \;, y_i + \frac{1}{2}\, hf_{i+\frac{1}{2}}\right) \end{aligned}\right\} \qquad (8.5a)$$

and

$$\left.\begin{aligned} \bar{y}(x_{i+1}) &= y_i + hf_{i+\frac{1}{2}} \\ f_{i+\frac{1}{2}} &\equiv f\left(x_i + \frac{1}{2}h \;, y_i + \frac{1}{2}\, hf_{i+\frac{1}{2}}\right) \end{aligned}\right\} \qquad (8.5b)$$

The schemes (8.5) are A – stable, one-stage Runge-Kutta methods with an error constant $c_3 = \frac{1}{24}$ and are of order two.

Case n = 2

From (8.3) and (8.4), we select the following equations

$$\bar{y}(x_i) = y_i\ ,\ I = 1$$

$$\bar{y}'(x_i) = f_1\ ,\ \bar{x}_1 = x_i + \left(\sqrt{3}+1\right)/\left(2\sqrt{3}\right)$$

$$\bar{y}'(\bar{x}_2) = f_2\ ;\ \bar{x}_2 = x_i + \left(\sqrt{3}+1\right)/\left(2\sqrt{3}\right),\ M = 2$$

to obtain a_0, a_1 and a_n in (8.2), where $\bar{x}_1$ and $\bar{x}_2$ are the Gaussian points in $[x_i\ ,\ x_{i+1}]$.

The resulting approximations after simplifications are

$$\left.\begin{aligned} &\bar{y} = y_i + (x - x_i)\left\{\beta_2(x)\ f_2 + \beta_1(x)\ f_1\right\},\quad x_i \le x \le x_{i+1} \\ &\beta_1(x) = \left\{(x - x_i) - \frac{1}{3}\left(3-\sqrt{3}\right)h\right\}\Big/\left(2h\sqrt{3}\right) \end{aligned}\right\} \tag{8.6a}$$

and

$$\left.\begin{aligned} &\bar{y}(x_{i+1}) = y_i + \frac{h}{2}(f_1 + f_2) \\ &f_1 = f\left(x_i + \left(\frac{1}{2} - \sqrt{3}/6\right)h,\ y_i + hf_2/4 + \left(\frac{1}{4} - \sqrt{3}/6\right)hf_2 + hf_1/4\right) \\ &f_2 = f\left(x_i + \left(\frac{1}{2} - \sqrt{3}/6\right)h,\ y_i + hf_2/4 + \left(\frac{1}{4} - \sqrt{3}/6\right)hf_1\right) \end{aligned}\right\} \tag{8.6b}$$

The schemes (8.6) are A-stable, order four, two-stage Runge-Kutta methods with an error constant $c_5 = \dfrac{1}{4320}$

In particular, (8.6b) is the Hammer and Hollingsworth formular (see Lambert, 1973). More collocation points at the Gaussian points [x_i , x_{i+1}] will lead to higher order methods in this case.

8.3 New Gaussian Point Methods

We consider collocation using combined off-grid and grid points

Case n = 1 ; k = 1

From (8.3) and (8.4), we select the following equations

$$\bar{y}(x_i) = y_i;\ I = 1,$$

$$\bar{y}(x_i) = f_i$$

$$\bar{y}'(x_{i+1/2}) = f_{i+1/2}$$

$$\bar{y}'(x_{i+1}) = f_{i+1}\ ;\ M = 3$$

to obtain a_0 , a_1 and a_3 in (8.2). The resulting approximations after simplification are

$$\bar{y}(x) = y_i + \frac{h^2}{3}\left[\beta_2(x)\ f_{i+1} + \beta_1(x)\ f_{i+1/2} + \beta_0(x)\ f_i\right]$$

$$\beta_0(x) = 2(x-x_i)^3 - 9h/2\,(x-x_i)^2 + 3h^2(x-x_i)$$

$$\beta_1(x) = -4(x-x_i)^3 + 6h(x-x_i)^2$$

(8.7a)

$$\beta_2(x) = 2(x-x_i)^3 - 3h/2\,(x-x_i)^2$$

and

$$\bar{y}(x_{i+1}) = y_i + \frac{h}{6}\left(f_i + 4 f_{i+\frac{1}{2}} + f_{i+1} \right)$$

(8.7b)

where (8.7b) is the well-known fourth order Gragg-Stetter scheme (Lambert, 1973), with an error constant

$$c_5 = \frac{1}{2880} ..$$

From equation (8.7b) we obtain f_{i+1} for the proposed continuous hybrid scheme (8.7a). To obtain the off-grid function value $f_{i+\frac{1}{2}}$ in (8.7), we use the following order three formula:

$$y_{i+\frac{1}{2}} = \frac{1}{2}(y_{i+1} + y_i) - \frac{h}{8}(f_{i+1} + f_i) \qquad (8.8)$$

suggested in England and Matthiej (1985). The combined schemes (8.7b) and (8.8) yield an order four A – stable method. (England and Matthiej, 1985).

Case n = 5, k = 2

From (8.3) and (8.4), we select the following equations

$$\bar{y}(x_{i+j}) = y_{i+j}; \quad j = 0,1, \ I = 2$$

$$\bar{y}'(x_i) = f_i$$

$$\bar{y}'(x_{i+1}) = f_{i+1}$$

$$\bar{y}'\left(x_{i+\frac{3}{2}} \right) = f_{i+\frac{3}{2}}$$

$$\bar{y}'\left(x_{i+\frac{3}{2}} \right) = f_{i+2} \ ; M = 4$$

to obtain a_0 , a_1, … a_5 in equation (8.2). The resulting approximation after simplification produces
the following Butcher scheme of order five

$$y_{(i+2)} = \frac{32}{3!}y_{n+1} - \frac{1}{3!}y_n + \frac{h}{93}\left[15 f_{n+2} + 64 f_{n+3/2} + 12 f_{n+1} - f_n\right] \quad (8.9)$$

with an error constant $c_6 = \dfrac{-1}{5580}$ (see Lambert, 1973).

8.4 Collocation at the Grid Points

Consider collocation at the grid points x_i , x_{i+1}, … , x_{i+k-1} , x_{i+k}. The constructed collocation polynomial approximations in this section are obtained and put in the form

$$\bar{y}(x) = \sum_{j=0}^{k-1} \alpha_i(x)\, y_{i+j} + (x - x_{i+k-1}) \sum_{j=0}^{k} \beta_j(x)\, f_{i+j} \qquad (8.10)$$

where α_j *and* β_j are specified polynomials of degree at most *k*. From equation (8.10) we produce many of the popular conventional *k*-step methods by using $y(x_{i+k})$. We now summarise some of such results.

8.4.1 The Adam-Moulton Methods

From (8.2), we get

$$\bar{y}(x_{i+k-1}) = y_{i+k-1} \ , \ I = 1, k \geq 1$$

$$\bar{y}'(x_{i+j}) = f_{i+j} \ ; \ j = 0(1)k, \qquad M = k+1$$

We solve the *(k + 2)* equations to give

$$\bar{y}(x) = y_{i+k-1} + (x - x_{i+k-1}) \sum_{j=0}^{k} \beta_j(x)\, f_{i+j}$$

Case k = 1

$$\bar{y}(x) = y_i + \frac{1}{2}(x - x_j)(x - x_i)\, f_j \qquad (8.11a)$$

$$\bar{y}(x_{i+1}) = y_i + \frac{1}{2}\left(f_{i+1} - f_i\right)$$

(8.11b)

where equation (8.11b) is the Trapezoidal rule of order two with an error constant $c_3 = -\frac{1}{12}$.

Case k = 2

$$\beta_0(x) = -(x - x_{i+1})/4h + (x - x_{i+1})^2/6h^2$$

$$\beta_1(x) = -(x - x_{i+1})^2/3h^2 + 1$$

$$\beta_2(x) = -(x - x_{i+1})^2/6h^2 + (x - x_{i+1})/4h$$

Thus,

$$\bar{y}(x_{i+2}) = y_{i+1} + \frac{h}{12}\left(5f_{i+2} + 8f_{i+1} - f_i\right) \qquad (8.12)$$

where equation (8.12) is the two step Adam-Moulton method of order three.

Case k = 3

$$\beta_0(x) = -\frac{1}{24h^3}(x - x_{i+2})^3 + \frac{1}{12h}(x - x_{i+2})$$

$$\beta_1(x) = \frac{1}{8h^3}(x - x_{i+2})^3 + \frac{1}{6h^2}(x - x_{i+2})^2 - \frac{1}{12h^2}(x - x_{i+2})$$

$$\beta_2(x) = \frac{1}{8h^3}(x - x_{i+2})^3 - \frac{1}{3}(x - x_{i+2})^2 - \frac{1}{4h}(x - x_{i+2}) + 1$$

$$\beta_3(x) = \frac{1}{24h^3}(x - x_{i+2})^3 + \frac{1}{6h^2}(x - x_{i+2})^2 + \frac{1}{6h}(x - x_{i+2})$$

Thus,

$$\bar{y}(x_{i+3}) = y_{i+2} + \frac{h}{24}\left(9f_{i+3} + 19f_{i+2} - 5f_{i+1} + f_i\right)$$
(8.13)

is the three-step Adam-Moulton method of order four. We remark that higher order members can be produced in a similar manner.

8.4.2 **Specific Equation (8.3) and (8.4) for Other Classes of Methods**

8.4.2.1 The Adam-Bashforth Methods

$$\bar{y}(x_{i+k-1}) = \bar{y}_{i+k-1} \;;\; l = 1,\; k \geq 1$$

$$\bar{y}(x_{i+j}) = f_{i+j} \;;\; j = 0, 1, \ldots, \quad k - 1 \;;\; M = k = n$$

8.4.2.2 The Optimal k-step Methods

$$\bar{y}(x_i) = y_i \;;\; l = 1,\; k \geq 1$$

$$\bar{y}(x_{i+j}) = f_{i+j} \;;\; I = 0, 1, \ldots, \quad k - 1 \;;\; M = k + 1 = n$$

8.4.2.3 The Backward Differentiation Formulae

$$\bar{y}(x_{i+j}) = y_{i+j} \;;\; j = 0, \ldots, \; k - 1 \;;\; I = k \geq 1$$

$$\bar{y}'(x_{i+k}) = f_{i+k} \;;\; M = 1,\; n = k$$

8.4.2.4 The Mid-Point Method, k = 2

$$\bar{y}(x_{i+j}) = y_{i+j} \;;\; j = 0, 1, \ldots, \; k - 1 \;;\; I = 2$$

$$\bar{y}'(x_{i+1}) = f_{i+k} \;;\; M = 1,\; n = 2$$

8.4.2.5 A New Class of Methods, k > 2

$$\bar{y}(x_{i+j}) = y_{i+j} \;;\quad j=0,\ k-1\;;\quad I=2$$

$$\bar{y}'(x_{i+j}) = f_{i+j} \;;\quad j=1(1)k,\quad M=k$$

The resulting methods are

Case k = 2

$$\bar{y}(x_{i+2}) = \frac{4}{5}y_{i+1} + \frac{1}{5}y_i + \frac{h}{5}(2f_{i+2} + 4f_{i+1}) \tag{8.13a}$$

Case k = 3

$$\bar{y}(x_{i+3}) = \frac{9}{8}y_{i+2} + \frac{1}{8}y_i + \frac{h}{8}(3f_{i+3} + 6f_{i+2} - 3f_{i+1}) \tag{8.13b}$$

Case k = 4

$$\bar{y}(x_{i+4}) = \frac{224}{243}y_{i+3} + \frac{19}{243}y_i + \frac{h}{243}(84f_{i+4} + 228f_{i+3} - 72f_{i+1} + 60f_{i+1}) \tag{8.13c}$$

Table 1: Order, Stability and Error Constants

Methods	Order	Absolute Stability Interval	Error Constant
Adam-Moulton New Methods (8.13a)	3 3	[-1,0] [-4,0]	- 1/24 - 1/30
Adam-Moulton New Methods (8.13b)	4 4	[-3,0] $\left[-2\tfrac{2}{3}, 0\right]$	$-19/720$ $-18/720$
Adam-Moulton New Methods (8.3)	5 5	[- 1.8,0] [- 1.6,0]	$-21/120$ $-21/1215$

Remark

The new methods (8.13a), (8.13b) and (8.13c) are compared with Adam-Moulton methods in Table 1 above. They have smaller error constants than the Adam-Moulton methods and have adequate stability intervals for non-stiff problems.

8.5 Derivative Approximations

Let

$$y(x) = \alpha_0 + a_1 x, \qquad x_i \le x \le x_{i+1}.$$

Then

$$y(x_i) = \alpha_0 + a_1 x_i = y_i$$

$$y(x_{i+l}) = a_0 + a_1 x_{i+l} = y_{i+l}.$$

Thus

$$y_{i+l} - y_i = a_l h$$

and so

$$a_1 = \frac{y_{i+1} - y_i}{h} = y'(x_i) \tag{8.14}$$

where equation (8.14) is the forward difference approximation to the first derivative.

Similarly, if we let

$$\bar{y}(x) = a_0 + a_1 x + a_2 x^2, \qquad x_{i-1} \le x \le x_{i+1}$$

then, from the equation

$$a_0 + a_1 x_i + a_2 x_i^2 = y_j = i-1,\ i, i+1$$

we get the central difference approximation

$$y''(x_i) = 2a_2 = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \tag{8.15}$$

$$y'(x_i) = a_1 + 2a_2 x_i$$

$$= \frac{y_{i+1} - y_{i-1}}{2h} \qquad (8.16)$$

8.6 A Special Second Order ODE

Let us consider

$$y''(x) = f(x, y) \qquad (8.17)$$

where $y(a)$ and $y''(a)$ are specified.

When $k = 2$ and we consider $n = 4$ in (8.2),

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \ , \ x_i \le x \le x_{i+2} \qquad (8.18)$$

From equation (8.3), we have

$$\left.\begin{aligned} \bar{y}(x_{i+j}) &= y_{i+j} \ ; \ j = 0,1 \\ \bar{y}'(x_{i+j}) &= f_{i+j} \ ; \ j = 1,3 \end{aligned}\right\} \qquad (8.19)$$

The remaining condition necessary to determine a_0, a_1, … , a_4 uniquely is given by $\bar{y}'(x_i) = f_i$

Thus, we obtain

$$\bar{y}(x) + \alpha_1(x) y_{i+1} + \alpha_0(x) y_i = \beta_2(x) f_{i+2} + \beta_1(x) f_{i+1} + \beta(x) f_i \qquad (8.20)$$

where

$$a_0(x) = \frac{x - x_{i+1}}{h}$$

$$a_1(x) = -\alpha_0(x) - 1$$

$$\beta_0(x) = \frac{1}{2} H_1(x) - \frac{1}{6} h H_2(x) + \beta_2(x)$$

$$\beta_1(x) = \frac{1}{6h} H_2(x) - 2\beta_2(x)$$

$$H_1(x) = (x - x_{i+1})^2 + h(x - x_{i+1})$$

Finally, at x_{i+2}, equation (8.20) becomes

$$\bar{y}(x_{i+2}) - 2y_{i+1} + y_i = \frac{h^2}{12}\left(f_{i+2} + 10 f_{i+1} + f_i\right) \quad (8.21)$$

where equation (8.21) is the well-known Numerov formula of order four with error constant $c_5 = -1/240$.

From this study, therefore, many important classes of finite difference methods were produced including new ones which are generally more accurate (with smaller error constants) than the Adams-Moulton methods and have adequate absolute stability intervals for non-stiff problems. The use of power series as basis function in the assumed trial solution was exploited. This work laid the foundation for research work in this area for several authors thereafter.

Adeniyi (1994) reported a combination of this idea and the tau method with the choice of canonical polynomials as basis function for the derivation of continuous forms of the Trapezoidal methods, the Simpson's method and the Gragg and Stetter one-step implicit hybrid method of order four.

Because of the elegant properties of the Chebyshev polynomials which I had earlier highlighted, their choice as basis functions in the assumed trial solution was made in Adeniyi and Alabi (2009) to construct continuous forms of some existing and new linear multistep methods for solution of first order initial value problems. The resulting schemes were accurate and effective. In a similar vein, Adeniyi and Alabi (2011) focused on the development of methods for direct solution of problems with higher orders without recourse to reduction of the equations to systems of first order which is the conventional approach. These methods performed favourably well in accuracy. A six-step method which emanated from this study has an order of eight with a very small error constant.

In Areo and Adeniyi (2013b), a self-starting LMM for direct solution of second order problems was reported while Mohammed and Adeniyi (2014a) obtained a three-step implicit hybrid LMM for problems of third order. The work in Mohammed and Adeniyi (2014c) was concerned with the construction of five-step block hybrid backward differentiation formulae for second order problems. These block methods simultaneously generated approximate solutions at different grid points in the interval of integration compared to the LMMs or Runge-Kutta methods, and are less expensive in terms of the number of function evaluations. They are also self-starting.

Recently, Adeniyi and Ekundayo (2014), Adeniyi and Taiwo (2015) and Adeniyi and Bamgbala (2015) constructed some orthogonal polynomials with different weight functions and over different intervals. These polynomials were exploited as basis functions in the trial solution (assumed approximation) to desired solutions of some IVPs. The resulting numerical schemes – block and non-block forms – were also consistent and zero stable (hence convergent). Numerical evidences arising from their practical implementations on some test problems also confirmed their accuracy and effectiveness in handling problems within their scope of coverage.

Our more recent works on this subject were reported in Ndukum *et al*. (2015) where the fourth order trigonometrically fitted method with the block unification implementation approach for oscillatory problems was developed, and Biala *et al*. (2015) which reported the derivation of block hybrid Simpson's method with two off-grid points for solution of stiff systems.

As an illustrative numerical example, consider here the application of the trapezoidal method

$$\bar{y}(x_{i+1}) = y_i + \frac{h}{2}\left(f_{i+1} - f_i\right) \tag{8.22}$$

whose continuous form is

$$Y(x) = Y_k + \frac{1}{2}(x - x_k)(f_k + f_{k+1}) \tag{8.23}$$

and the Gragg-Stetter method

$$\bar{y}(x_{i+1}) = y_i + \frac{h}{6}\left(f_i + 4 f_{i+1/2} + f_{i+1} \right) \qquad (8.24)$$

whose continuous form is

$$\bar{y}(x) = y_i + \frac{h^2}{3}\left[\beta_2(x)\ f_{i+1} + \beta_1(x)\ f_{i+1/2} + \beta_0(x)\ f_i \right] \qquad (8.25)$$

$$\beta_0(x) = 2(x - x_i)^3 - 9h/2\,(x - x_i)^2 + 3h^2(x - x_i)$$

$$\beta_1(x) = -4(x - x_i)^3 + 6h(x - x_i)^2 \qquad (8.26)$$

$$\beta_2(x) = 2(x - x_i)^3 - 3h/2\,(x - x_i)^2$$

to the nonlinear IVP

$$y'(x) = 1 + y^2,\ 0 \le x \le \pi/4\ ,\ y(0) = 0 \qquad (8.27)$$

whose analytic solution y(x) = tan x, is smooth. The results for $h = 0.5\pi/4$ *and* $h = 0.1\pi/4$ are compared in Tables 8.2(a) and 8.2(b).

Table 2(a) Error of Methods for Problem (8.27) with the step length h = $0.5\pi/4$, δx = h/10

X	Order Two Methods		Order Four Methods	
	Trapezoidal Method (8.22)	Continuous Scheme (8.23)	Gragg and Setter Method (8.24)	Continuous Scheme (8.25)
0.0	0.0	0.0	0.0	0.0
0.927E-2		3.59010E-3		8.95976E-4
0.854E-2		7.05872E-3		1.23950E-3
0.1178		1.02828E-2		1.15357E-3
0.1571		1.31364E-2		7.64247E-4
0.1963		1.54887E-2		2.02264E-4
0.2356		1.72025E-2		3.95200E-4
0.2749		1.81323E-2		8.82592E-4
0.3142		1.81200E-2		1.10381E-3
0.3534		1.70024E-2		8.89718E-4
0.3927	1.45886E-2	1.45886E-2	5.5268E-3	5.52684E-3
0.4320		3.40708E-2		2.80864E-3
0.4712		5.18266E-2		2.14655E-3
0.5105		6.7001E-2		2.21442E-3
0.5498		8.11012E-2		3.33235E-3
0.5890		9.19983E-2		3.74187E-3
0.6283		9.99094E-2		3.49408E-3
0.6676		1.04390E-1		1.73410E-3
0.7069		1.04921E-1		2.68529E-3
0.7461		1.00886E-2		2.32329E-3
0.7854	9.15518E-2	9.15518E-2	8.60412E-5	3.60412E-5

Table 2(b) Error of Methods for Problem (8.27) with h = $0.1\pi/4$, δx = h/10

X	Order Two Methods		Order Four Methods	
	Trapezoidal Method (8.22)	Continuous Scheme (8.23)	Gragg and Setter Method (8.24)	Continuous Scheme (8.25)
0.0	0.0	0.0	0.0	0.0
7.8540E-3		2.42127E-5		5.86204E-6
1.5708E-2		4.74564E-5		7.82843E-6
2.3562E-2		6.87615E-4		6.86870E-6
3.1416E-2		8.71575E-4		3.95333E-6
3.9269E-2		1.01672E-4		5.43369E-6
4.7124E-2		1.11332E-4		3.85450E-6
5.4978E-2		1.15161E-4		6.79694E-6
6.2832E-2		1.12180E-4		7.79694E-6
7.0686E-2		1.01405E-4		5.86244E-6
7.8540E-2	8.18519E-5	8.18519E-5	1.607765E-8	1.66765E-8
8.6397E-2		1.5125E-4		6.15674E-6
9.4248E-2		2.0988E-4		8.25792E-6
1.0210E-1		2.56749E-4		7.28815E-6
1.0996E-1		2.90842E-4		4.25446E-6
1.1781E-1		3.11149E-4		1.70054E-7
1.2566E-1		3.16648E-4		3.94502E-6
1.3352E-1		3.06313E-4		7.0627E-6
1.4137E-1		2.79108E-4		8.1476E-6
1.4922E-1		2.33988E-4		6.15552E-6
1.5708E-1	1.690000E-4	1.69000E-4	3.30620E-8	3.30620E-8

Remark: Continuous schemes generated more solution (output) than their discrete equivalents.

Conclusion

Mr. Vice-Chancellor Sir, in the course of this lecture, I have presented a fast, efficient and reliable error estimation technique for a numerical method for the solution of ordinary differential equations. The specific numerical method is the tau method, which was originally developed to solve linear problems with polynomials coefficients and whose scope of application has been extended to non-linear problems, non-polynomial coefficients problems, partial differential equations, integral equations and integro-differential equations. The three variants of the tau method namely the differential form, the integral form and the recursive form have been considered and the error estimation for each of the variants have been discussed. For all the three, the resulting estimates obtained accurately capture the order of the tau approximation to the analytic solution, thus justifying the desirability of the technique. The extension of this method of error analysis of the tau method to non-linear problems, piece-wise solution, discrete formulations for ODEs and partial differential equations confirm that the error estimation is good in terms of accuracy and effectiveness.

The development of continuous forms of existing and new linear multistep methods as well as hybrid methods for direct and indirect solution of initial and boundary value problems has also been presented. The main attractions of these continuous numerical integration schemes are their ability to yield several output of solutions at the off-grid points without requiring additional interpolation and at no extra cost. These render the methods efficient, accurate and highly desirable.

Recommendations

Mr. Vice-Chancellor Sir, much has been said in this Lecture on the minimisation of the error of a function which is implicitly defined by a differential equation and for which the Chebyshev polynomial is a major factor. The mini-max property and the equi-oscillation of these polynomials which also lead to even distribution of error when used for function approximation in the entire range of its definition account for this.

For some numerical methods, the introduction of round-off error at any stage of their implementation may not affect the final output significantly, in which case the error either fizzles out or does not grow. Such schemes are stable. If on the other hand the error introduced adversely affects the final output as to render it unacceptable, in which case there is much deviation from the expected result, the scheme is unstable.

All deviations from acceptable, ideal standards and norms of any just and egalitarian society are necessarily errors. When errors are minimised in a system whether mathematical, social, political, such systems experience relative stability and progress. Nigeria is a country where the errors in our systems have led to injustice, in-equity, disorderliness and sometimes outright crisis and chaos.

Before I humbly submit my recommendations, we need to give greater attention to the needed systemic Chebyshev polynomials that will bring about stability, peace and progress in our nation. And what are they? They are love for our fellow humans, tolerance and respect for human life and more importantly the Fear of GOD.

Other recommendations are:

1. Judicious allocation of resources to the various sectors of our economy to minimise errors resulting to wastage of scarce resources.
2. Proper monitoring of budget implementation to minimise the errors of corruption which in recent times have resulted to large scale embezzlement.
3. Proper monitoring of structural buildings to forestall the error that may result to collapse of buildings.
4. Allowance for use of **non-programmabl**e calculators for examination purposes at all levels in this University as was the practice before, in order to avoid the errors resulting from brain fatigue. In appreciation of developing science and technology, some examination bodies such as WAEC allow the use of calculators. WAEC goes as far as supplying her examination candidates with calculators with functions specifically allowed for the examinations. The University can as well borrow a leaf from this by giving out non-transferable customised but affordable calculators to all students,

particularly those of Science and Engineering. This will invariably soothe their pains and earn substantial revenue..

5. The use of Numerical methods such as interpolation and extrapolation techniques for projecting our population at national, state and local government levels. This will greatly reduce waste of scarce resources that could be used for other profitable ventures/projects.
6. Greater allocation of resources to research with less stringent conditions to avoid the error of deprivation of researchers who sometimes may not be able to articulate their proposals well enough.
7. Reduction of number of courses examinable by CBT in the University especially at higher levels so as to minimize the error which may result to the production of graduates who are not able to write good English.

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